Continuous Fourier Transform

- We have introduced the continuous Fourier transform, the most general form of Fourier transform.
- The continuous Fourier transform defines completely and exactly the frequency domain, where the frequency domain is continuous and range un-limited.
Finite Duration Signal and Band Limited Signal

- **Finite Duration Signal**: A signal $x(t)$ is nonzero in $[-t_b, t_b]$ for some $t_b > 0$, and is zero elsewhere.
- **Band Limited Signal**: A signal’s frequency is nonzero in the frequency band $[-w_b, w_b]$ for some $w_b > 0$, and is zero elsewhere.

<table>
<thead>
<tr>
<th>Finite duration in time domain</th>
<th>Band unlimited in frequency domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Band limited in frequency domain</td>
<td>Infinite duration in time domain</td>
</tr>
</tbody>
</table>
Finite Duration Signal and Band Limited Signal

- Because a finite-duration signal can be represented as the multiplication of some signal of a rectangular window.

- Time domain multiplication, frequency domain is the convolution of the sync function.

- Since the sync function is band unlimited. The convolution of it and any function is band unlimited.

**Figure 11-4:** Fourier transform of a rectangular pulse. (a) Time function $x(t) = u(t + \frac{1}{2}T) - u(t - \frac{1}{2}T)$, and (b) Corresponding Fourier transform $X(j\omega)$ is a sinc function.
Similarly, a band-limited signal can be represented as the multiplication of some spectrum of a rectangular window in frequency domain.

Frequency domain multiplication, time domain is the convolution with the sync function.

Since the sync function is infinite durational. The convolution of it and any function is infinite durational.

Figure 11-5: Fourier transform of sinc function: (a) Time function \( x(t) = \frac{\sin(\omega_b t)}{\pi t} \), and (b) corresponding Fourier transform \( X(j\omega) = u(\omega + \omega_b) - u(\omega - \omega_b) \).
Discrete-time Signals in Continuous Domain

• How to represent a discrete-time signal in the time domain for continuous Fourier transform?

• A discrete-time signal can be represented as a sequence of impulse functions (an impulse train) occurred at equally spaced time instances, in the continuous-functional domain.
Discrete-time Signals

• A common way to obtain a discrete-time signal is to sample a continuous-time signal at equally spaced time instances.

• Referred to as uniform sampling:
Continuous Fourier Transform of an Impulse Sequence

Recall

• When the frequency domain spectrum is an equally-spaced impulse sequence, what is the time domain signal?

• The time-domain signal is a periodic function.
Continuous Fourier Transform of an Impulse Sequence

Remember that

- Continuous Fourier transform is time-frequency analogous.

Forward

\[ F(jw) = \int_{-\infty}^{\infty} f(t)e^{-jwt} \, dt \]

Backward

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(jw)e^{jwt} \, dw \]
Continuous Fourier Transform of an Impulse Sequence

• If we (purposively) perform forward transform for the complex conjugate of the frequency domain spectrum $F^*(jw)$ and exchange the roles of $w$ and $t$, we have the following symmetry property of the Fourier transform:

$$\int_{-\infty}^{\infty} F^*(jw) e^{-jwt} \, dw = \left( \int_{-\infty}^{\infty} F(jw) e^{jwt} \, dw \right)^* = 2\pi f^*(t)$$

Forward transform of $F^*(jw)$

apply the rule of

Backward transform
Continuous Fourier Transform of an Impulse Sequence

• Remember that when $F(jw)$ is an equally-spaced impulse sequence, $f(t)$ is a periodic function.

• By exchanging the variables $w$ and $t$, we have

$$\int_{-\infty}^{\infty} F^*(jt) e^{-jwt} dt = 2\pi f^*(w)$$

• That is, when $F^*(jt)$ is a time-domain signal, its frequency domain spectrum is $2\pi f^*(w)$. 
Continuous Fourier Transform of an Impulse Sequence

• Since taking the complex conjugate of \( f(t) \) or \( F(jw) \) does not affect the properties of being a periodic function or an impulse sequence.

• So, when the time domain signal is an equally-spaced impulse sequence, the frequency domain spectrum is a periodic function.
Continuous Fourier Transform of an Impulse Sequence

- The spectrum is a periodic function: Hence, the frequency components appear repeatedly from low frequency bands to very high (unlimited) frequency bands.
- It means that a discrete-time signal always has very high frequency components, since the spectrum is repeated (toward infinity).
- This is because that impulse signals can be treated as a high-oscillation in a very short time interval.
Sampling a Continuous Function

- The property that a discrete-time signal always has very high frequency components could violate our intuition.
- It is because that it is quite often the situation that a discrete-time signal are samples from a relatively ‘smoother’ continuous-time signals.
- What is the relationship between the spectra of the continuous-time signal and the sampled signal?
Uniform Samples of a Continuous Function

• To proceed, let us first consider the uniform samples of a basis function, $e^{jwt}$.
  – When varying $w$, we obtain infinite (uncountable) basis functions of different frequencies.

• Assume that $e^{jwt}$ is uniformly sampled with the time step $T$ and becomes $e^{jwnT}$ ($n$ is an integer). We also obtain infinite (but countable) basis functions.

• What is the difference between the set of bases $e^{jwt}$ ($t \in \mathbb{R}$) and its samples $e^{jwnT}$ ($n \in \mathbb{Z}$)?
Sampling a Continuous Function

• **Property**: the bases set $e^{jwnT}$ is a periodic function in $w$ with the common period $2\pi/T$ for all $n \in \mathbb{Z}$. (because $e^{jwnT} = e^{j(w+2\pi/T)nT}$)

• Note that the bases set $e^{jwt}$ does not have this property in general when $t \in \mathbb{R}$; in other words, this property holds only when we perform sampling with a period $T$ (and so when $t$ are multiples of a common period $T$).
  
  – For every $e^{jwt}$, $2\pi/t$ is a period for $t \in \mathbb{R}$ in the $w$ domain, but there is no common period.
Sampling a Continuous Function

• Now, let us assume that an arbitrary discrete-time signal \( x_a(nT) \) (\( n \) is an integer) is uniformly sampled from the analog signal \( x_a(t) \) with the time step \( T \).

• Apply the inverse Fourier transform to any of the time-domain samples \( x_a(nT) \), we have

\[
x_a(nT) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\omega) e^{j\omega(nT)} d\omega
\]
Sampling a Continuous Function

• We would like to employ the property that $e^{j\omega nT}$ is a periodic function in $\omega$ with period $2\pi/T$, and see what happens:

• Let’s express the above equation, which involves integration from $\omega=-\infty$ to $\omega=\infty$, as the sum of integrals over successive intervals each equal to one period $2\pi/T = \omega_s$.

\[
x_a(nT) = \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} \int_{\frac{(2r+1)\pi}{T}}^{\frac{(2r+1)\pi}{T}} X_a(j\omega) e^{j\omega nT} d\omega
\]
Sampling a Continuous Function

• Each term in this summation can be reduced to an integral over the range \(-\pi/T\) to \(\pi/T\) by dividing the range of \(w\)

\[
x_a(nT) = \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} \int_{-\pi/T}^{\pi/T} X_a \left( jw + j \frac{2\pi r}{T} \right) e^{j\omega nT} e^{j2\pi rn} dw
\]

\[
x_a(nT) = \sum_{r=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} X_a \left( jw + j \frac{2\pi r}{T} \right) e^{j\omega nT} dw
\]

= 1
Spectrum of the Discrete-time Signal

Sampled signals

- As mentioned, the discrete-time samples in the continuous function domain is represented as an impulse sequence,

\[ x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT) \delta(t - nT) \]

- From now on, we focus on the Fourier transform of the discrete-time signal, and see how it is related to the analog spectrum \( X_a(jw) \)
Spectrum of the Discrete-time Signal

• The Fourier transform of the discrete-time samples \( x_s(t) \) is

\[
X_s(jw) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x_a(nT) \delta(t - nT) e^{-jwt} \, dt
\]

\[
= \sum_{n=-\infty}^{\infty} x_a(nT) \int_{-\infty}^{\infty} \delta(t - nT) e^{-jwt} \, dt
\]

\[
= \sum_{n=-\infty}^{\infty} x_a(nT) e^{-jwnT}
\]

Fourier transform of a shifted impulse function
Spectrum of the Discrete-time Signal

- Unlike $x_a(nT)$, it can be seen from the above that $X_s(jw)$ is periodic in $w$ with the period $2\pi/T$.
- Hence, $X_s(jw)$ can be represented by Fourier series,

$$X_s(jw) = \sum_{n=-\infty}^{\infty} c_n e^{jnTw} = \sum_{n=-\infty}^{\infty} c_{-n} e^{j(-n)Tw}$$

and the Fourier series coefficient $c_n$ is

$$c_n = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} X_s(jw)e^{-jnTw}dw$$
c.f. Recall of the Fourier Series Theory

Forward Transform

\[ a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t)e^{-jk\omega_0 t} \, dt \]

Inverse Transform

\[ x_{T_0}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

• where \( \omega_0 = \frac{2\pi}{T_0} \)
Uniformly Sampling a Continuous Function

• Compared the result derived from the definition

\[ X_s(jw) = \sum_{n=-\infty}^{\infty} x_a(nT)e^{-jwnT} \]

to the result from Fourier series

\[ X_s(jw) = \sum_{n=-\infty}^{\infty} c_{-n}e^{-jnTw} \]

• We have

\[ x_a(nT) = c_{-n} = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X_s(jw)e^{jwnT} \, dw \]
Uniformly Sampling a Continuous Function

Then, from both results of \( x_a(nT) \),

\[
x_a(nT) = \sum_{r=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} X_a \left( jw + j \frac{2\pi r}{T} \right) e^{jwnT} dw
\]

and

\[
x_a(nT) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} X_s(jw) e^{jwnT} dw
\]

We have

\[
X_s(jw) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_a \left( jw + j \frac{2\pi r}{T} \right)
\]
Uniformly Sampling a Continuous Function

\[ X_s(jw) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_a\left(jw + j \frac{2\pi r}{T}\right) \]

• **What does it mean?**

• Remember that \(X_a(jw)\) is the spectrum of the analog signal \(x_a(t)\)

• \(X_s(jw)\) is the spectrum of the uniformly sampled signal \(x_s(t)\)

• The equation means that \(X_s(jw)\) is a periodic duplication of the continuous Fourier transform \(X_a(jw)\) a period \(2\pi/T = w_s\) and scaled by \(T\).
Figure 3.2  The discrete-time signal $x_a(nT)$ obtained from the analog signal $x_a(t)$ and the discrete-time Fourier transform $H(e^{j\omega})$. 

An analog signal $x_a(t)$ and the magnitude of its Fourier transform $X(j\omega)$. 

$$|X_a(j\omega)|$$

(b)

$$|X(e^{j\omega})|$$

$$\frac{|X(j\omega)|}{T}$$

$$\frac{|X(j(\omega-\omega_s))|}{T}$$

$$\frac{|X(j(\omega-2\omega_s))|}{T}$$

0 $\omega_s$ $2\omega_s$
Aliasing Effect

• Because of the overlapping effect, more commonly known as “aliasing,” there is no way of retrieving $X_a(jw)$ from $X_s(jw)$

• In other words, we have lost the information contained in the analog function $x_a(t)$ if the aliasing occurs when we perform uniform sampling on it.
Aliasing Effect

• How to avoid aliasing?

• Band-limited signal: the continuous signal $x_a(t)$ is band limited if its Fourier transform $X_a(j\omega) = 0$ for $|\omega| > \omega_b$, where $\omega_b$ is a frequency bound.

• For a signal that is not band-limited, there is no chance to avoid aliasing.
Illustration for Avoid Aliasing

A case without aliasing:
Illustration for Avoid Aliasing

- In this case, aliasing occurs:
Avoid Aliasing

• From the above illustrations, we can see that
• If the continuous signal $x_a(t)$ is a band-limited signal with the bound $w_b$, then, the aliasing can be avoided when the sampling frequency is chosen such that $w_s > 2w_b$.
• When there is no aliasing, we can reconstruct the continuous signal from its uniform samples.
Sampling Theorem

- To avoid the situation of aliasing, the sampling frequency shall be larger than twice of the highest frequency of the continuous signal.

- Nyquist sampling theorem: $f_b (= w_b / 2\pi)$ is called the Nyquist frequency, and $2f_b$ is called the Nyquist rate.
Further illustration

Continuous signal

Sampled signal

The magnitude spectrum of the continuous signal

The spectrum of the sampled signal
Time-domain Example of Aliasing

• In the above, the aliasing is analyzed in frequency domain. What does it happen in time domain?

• From the sampling theorem, we see that aliasing occurs when the sampling rate $w_s$ is not high enough.

• For example, when we sample a sinusoidal signal with a low frequency, we can see the aliasing effect in time domain.
Example: sampling with sufficiently large rate.

**Figure 4-3:** A continuous-time 100-Hz sinusoid (top) and two discrete-time sinusoids formed by sampling at $f_s = 2000$ samples/sec (middle) and at $f_s = 500$ samples/sec (bottom).
Eg., sampling of insufficient rate: a 100 Hz sinusoid at $f_s = 80 \text{ samples/sec}$. 

![Diagram showing spectrum of the 100 Hz cosine wave with sampling at $T_s = 12.5 \text{ msec}$]
Sampling and aliasing

Sampled at frequency $f_s$, the function $\cos(2\pi tf)$ cannot be distinguished from $\cos[2\pi t(kf_s \pm f)]$ for any $k \in \mathbb{Z}$. 
Another way to Derive Sampling Theorem – from Convolution

- Recall that the sampled signal is of the form:

\[ x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t - nT) \]

- It can also be represented as the multiplication of the continuous-time signal \( x_a(t) \) and the impulse train signal

\[ s(t) = \sum_{-\infty}^{\infty} \delta(t - nT) \]

- That is \( x_s(t) = x_a(t) \cdot s(t) = x_a(t) \sum_{-\infty}^{\infty} \delta(t - nT) \)
Impulse train signal

\[ s(t) \]

\(-2t_s\) \(-t_s\) \(0\) \(t_s\) \(2t_s\) \(t\)
Examples of $x_s(t)$ for two sampling rates

$T = T_1$

$T = 2T_1$
Using Convolution

• Let us now consider the continuous Fourier transform of $x_s(t)$.

• Since $x_s(t)$ is a product of $x_a(t)$ and $s(t)$, its continuous Fourier transform is the convolution of $X_a(jw)$ and $S(jw)$.

• Property: the continuous Fourier transform of a periodic impulse train $s(t)$ is still a periodic impulse train.
where $w_s = 2\pi / T$ (or $f_s = 1 / T$) is the sampling frequency in radians/s.
Recall

Figure 11-10: Periodic impulse train: (a) Time-domain signal \( p(t) \); and (b) Fourier transform \( P(j\omega) \). Regular spacing in the frequency-domain is \( \omega_s = \frac{2\pi}{T_s} \).
Since \( S(jw) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(w - kw_s) \)

If follows that

\[
X_s(jw) = X_a(jw) \ast S(jw)
\]

Again, we see that the copies of \( X_a(jw) \) are shifted by integer multiples of the sampling frequency, and then superimposed to produce the periodic Fourier transform of the sampled signal.

\[
X_s(jw) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(j(w - kw_s))
\]
Frequency domain convolution

\[ X_c(j\Omega) \]

\[ \Omega \equiv \omega \]

(a)

\[ S(j\Omega) \]

(b)

\[ \frac{2\pi}{T} \]

\[ -2\Omega_s \quad -\Omega_s \quad 0 \quad \Omega_s \quad 2\Omega_s \quad 3\Omega_s \quad \Omega \]

\[ X_s(j\Omega) \]

(c)

\[ \Omega_N \quad \Omega_s \quad 2\Omega_s \quad 3\Omega_s \quad \Omega \]

\[ \Omega_N \quad \Omega_s \quad 2\Omega_s \quad 3\Omega_s \quad \Omega \]

\[ \Omega_N \quad \Omega_s \quad 2\Omega_s \quad 3\Omega_s \quad \Omega \]
Figure 4.3  Effect in the frequency domain of sampling in the time domain.
(a) Spectrum of the original signal.
(b) Spectrum of the sampling function.
(c) Spectrum of the sampled signal with $\Omega_s > 2\Omega_N$.  (d) Spectrum of the sampled signal with $\Omega_s < 2\Omega_N$. 
Reconstruct the continuous-time signal from the sampled signal

- If a band-limited signal is sampled with a frequency higher than the Nyquist rate, then it is possible to reconstruct the original continuous-time signal.

- Ideal low-pass filter: a system for which the frequency response is unity over a low range of frequencies and is zero at the high frequencies.
Ideal low-pass filter: frequency domain

- In frequency domain: the effect is equivalent to multiply the spectrum of the input signal and the rectangular window of the ideal low-pass filter.
\[ s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

Figure 4.4  Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.
Reconstruction and Anti-aliasing filter

• For a band-limited signal, if we sample it with a sufficient high rate (higher than the Nyquist rate), then it can be recovered exactly by ideal low-pass filter.

• Ideal low pass filter can also serve as an anti-aliasing filter when the signal is not band-limited (which is common in practice).

• In this way, some rear frequencies of the signal’s spectrum will be discarded or sacrificed, but the aliasing effect is avoided.
Nyquist limit and anti-aliasing filters

Without anti-aliasing filter

\[ X(f) \]

\[
\text{single-sided bandwidth}
\]

\[
\text{double-sided bandwidth}
\]

\[ \hat{X}(f) \]

\[-2f_s \quad -f_s \quad 0 \quad f_s \quad 2f_s \]

\[ f \]

With anti-aliasing filter

\[ X(f) \]

\[
\text{Nyquist limit} = \frac{f_s}{2}
\]

\[
\text{anti-aliasing filter}
\]

\[ \hat{X}(f) \]

\[ \text{reconstruction filter} \]

\[-2f_s \quad -f_s \quad 0 \quad f_s \quad 2f_s \]

\[ f \]

Anti-aliasing and reconstruction filters both suppress frequencies outside \(|f| < f_s/2\).
Ideal low-pass filter: time domain

• In frequency domain, multiplication of a rectangle window is performed for an ideal low-pass filter.
• In time domain, convolution is performed.
• Recall the Fourier transform of a rectangular function is a sync function.
Figure 12-31: Ideal bandlimited reconstruction filter: (a) Impulse response $h_r(t)$, and (b) frequency response $H_r(j\omega)$. 
Sync Function Interpolation

• Hence, in time domain, ideal low-pass filter acts as convolution with the sync function.
Reconstruction filter example

- **Red circles** represent the sampled signal.
- **Black line** is the interpolation result.
- **Blue lines** show scaled/shifted \( \sin(x)/x \) pulses.
Reconstruction filters

The mathematically ideal form of a reconstruction filter for suppressing aliasing frequencies interpolates the sampled signal $x_n = x(t_s \cdot n)$ back into the continuous waveform

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \cdot \frac{\sin \pi(t - t_s \cdot n)}{\pi(t - t_s \cdot n)}.$$
Discrete-time signals

• Remember that the spectrum of a discrete-time signal is periodic, which always has very high frequency.

• However, if we assume that the discrete-time signal is sampled from a continuous function following the sampling theorem, we can use the sync function to interpolate the continuous-time signal, and the spectrum is band-limited.

• In this way, the discrete-time signal can be imaged as samples from a “smooth” function.
Choice of sampling frequency

Due to causality and economic constraints, practical analog filters can only approximate such an ideal low-pass filter. Instead of a sharp transition between the “pass band” ($< f_s/2$) and the “stop band” ($> f_s/2$), they feature a “transition band” in which their signal attenuation gradually increases.

The sampling frequency is therefore usually chosen somewhat higher than twice the highest frequency of interest in the continuous signal (e.g., 4×). On the other hand, the higher the sampling frequency, the higher are CPU, power and memory requirements. Therefore, the choice of sampling frequency is a tradeoff between signal quality, analog filter cost and digital subsystem expenses.
Discrete-time Fourier Transform

• We have seen that a set of discrete samples can be represented as a series of impulses in the continuous domain. The Fourier transform of such a signal is a periodic function with the period $2\pi/T$.

• The spectrum is only informative within a single period $[-\pi/T, \pi/T]$ because others are repetitive or can be ignored by the smoothness assumption.

• In particular, there could also be signals that are discrete-time by nature (not necessarily samples of a continuous function)

• Therefore, the discrete-time Fourier transform (DTFT) is introduced for the spectrum representation the discrete-time signals, just like that Fourier series is particular for periodic function.
Discrete-time Fourier Transform

• Remember that in the derivation of uniformly sampling of continuous functions, we have shown that the discrete-time signal’s spectrum can be computed by the following summation instead of integration:

\[
X_s(jw) = \sum_{n=-\infty}^{\infty} x_a(nT)e^{-jwnT}
\]
The DTFT pair is defined as follows: Let \( x(n) \) be a discrete-time signal, \( n \in \mathbb{Z} \).

**Forward DTFT:**
\[
X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n) e^{-j\omega n}
\]

**Inverse DTFT:**
\[
x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega
\]

Note that we do not incorporate the sampling period \( T \) in the transform. It can be treated as the normalized case of \( T=1 \).
DTFT

• In digital signal processing, discrete-time signals are of the main interest, and DTFT is a main tool for analyzing such a signal.

• Since the frequency spectrum repeats periodically with the period $2\pi$, we usually consider only a finite-duration $[-\pi, \pi]$.

• We can bear in mind that, if we have the knowledge that the discrete-time signal is sampled from an analog signal by sampling theorem, then the DTFT spectrum is exactly the analog spectrum cut by an ideal low-pass filter.
DTFT

- When the discrete-time signal are sampled from a continuous function and knowing the sampling period $T$, the DTFT frequency $\pi$ corresponds to the analog frequency $\pi/T$.
- **High frequency region**: The frequency nearing $-\pi$ or $\pi$.
- **Low frequency region**: The frequency nearing $0$.
- DTFT is suitable to model the frequency response of discrete-time systems; will be introduced later.
Discrete-time Signals and Systems

• Up to now, we have introduced continuous-time signal representations in the frequency domain.
• We begin to introduce discrete-time signals and systems, which is rather simple in comparison to the continuous-time signals. In particular, the impulse function is straightforward and simple in the discrete-time domain.
• In the following, the “signals” are referred to as discrete-time signals, represented as the form like $x[n]$, $n$ is an integer.
A Unit Signal

• Unit sample sequence
  – Unit impulse function, Dirac delta function, impulse

\[ \delta[n] = \begin{cases} 
0 & n \neq 0 \\
1 & n = 0 
\end{cases} \]
An arbitrary sequence can be represented as a sum of scaled, delayed impulses.

\[
x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]
\]
Some Signal Examples

- Unit step sequence

\[ u[n] = \begin{cases} 
1 & n \geq 0 \\
0 & n < 0 
\end{cases} \]
Some Signal Examples (cont.)

- Real exponential sequence

\[ x[n] = A \alpha^n \]

\[
 y[n] = \begin{cases} 
 A \alpha^n & n \geq 0 \\
 0 & n < 0 
\end{cases} 
\]

- \( y[n] \) can be represented as

\[ y[n] = A \alpha^n u[n] \]
Some Signal Examples (cont.)

- Sinusoidal sequence

\[ x[n] = A \cos (w_0 n + \phi) \]
Recall the concept of **System**

- The set consisting of all signals (of a type) forms a functional space.
- **Signal Processing System**: map an input signal to an output signal
  - Continuous-time systems
    - Systems for which both input and output are continuous-time signals
  - Digital system
    - Both input and output are digital signals

\[
\begin{align*}
x[n] & \quad \rightarrow \quad T\{\cdot\} \quad \rightarrow \quad y[n] \\
\end{align*}
\]

- **System**: a function (or mapping) whose input and output are both functions.
Discrete-time Systems

- A transformation or operator that maps an input sequence with values $x[n]$ into an output sequence with value $y[n]$.

$$y[n] = T\{x[n]\}$$

- Eg., the ideal low-pass filter introduced before is a system.
System Examples

- **Ideal Delay**
  - $y[n] = x[n - n_d]$, where $n_d$ is a fixed positive integer called the delay of the system.

- **Moving Average**
  - $y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n - k]$

- **Memoryless Systems**
  - The output $y[n]$ at every value of $n$ depends only on the input $x[n]$, at the same value of $n$.
  - Eg. $y[n] = (x[n])^2$, for each value of $n$. 
System Examples (continue)

- Linear System: If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs.
- The system is linear if and only if
  - $T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$. 
  - $T\{ax[n]\} = aT\{x[n]\} = ay[n]$, for arbitrary constant $a$.
  - So, if $x[n] = \sum_k a_k x_k[n]$, $y[n] = \sum_k a_k y_k[n]$ (superposition principle)
- EG., Accumulator System

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$
(is a linear system)
System Examples (continue)

- **Nonlinear System.**
  - Eg. $w[n] = \log_{10}(|x[n]|)$ is not linear.

- **Time-invariant System:**
  - If $y[n] = T\{x[n]\}$, then $y[n-n_0] = T\{x[n-n_0]\}$
  - The accumulator is a time-invariant system.

- **The compressor system (not time-invariant)**
  - $y[n] = x[Mn]$, $-\infty < n < \infty$. 
System Examples (continue)

- Causality (the outputs depend only on the previous and current inputs)
  - A system is causal if, for every choice of $n_0$, the output sequence value at the index $n = n_0$ depends only the input sequence values for $n \leq n_0$.
  - That is, if $x_1[n] = x_2[n]$ for $n \leq n_0$, then $y_1[n] = y_2[n]$ for $n \leq n_0$.

- Eg. Forward-difference system (non causal)
  - $y[n] = x[n+1] - x[n]$ (The current value of the output depends on a future value of the input)

- Eg. Background-difference (causal)
  - $y[n] = x[n] - x[n-1]$