Chirp Transform for FFT

- Since the FFT is an implementation of the DFT, it provides a frequency resolution of $2\pi/N$, where $N$ is the length of the input sequence.
- If this resolution is not sufficient in a given application, we have the option of zero padding the input sequence.
- However, this may be unduly expensive in operation count if the required resolution is much higher than $2\pi/N$. 
In practice, one is seldom interested in high resolution over the entire frequency band. More often one is interested only a relatively small part of the band.

For example, suppose we wish to determine the frequency $\theta_0$ of a sinusoid to a good accuracy from $N$ data points. Performing FFT on the data will enable determination of $\theta_0$. If we could compute the DTFT $X_f(\theta)$ at high resolution in an interval of width $2\pi/N$ around the maximum point of the DFT, we might be able to improve the accuracy of $\theta_0$. A special technique, called chirp Fourier transform, accomplishes this.
From Oppenheim’s book *Chirp Transform Algorithm (CTA)*

This algorithm is not optimal in minimizing any measure of computational complexity, but it has been used to compute any set of equally spaced samples of the DTFT on the unit circle.

To derive the CTA, we let $x[n]$ denote an $N$-point sequence and $X(e^{j\omega})$ its DTFT. We consider the evaluation of $M$ samples of $X(e^{j\omega})$ that are equally spaced in angle on the unit cycle, at frequencies

$$w_k = w_0 + k\Delta w$$

$$\left(k = 0, 1, \ldots, M - 1, \right)$$

$$\Delta w = \frac{2\pi}{M}$$
When \( w_0 = 0 \) and \( M = N \), we obtain the special case, DFT.

The DTFT values evaluated at \( w_k \) (\( k = 0, \ldots, M-1 \)) are

\[
X(e^{jw_k}) = \sum_{n=0}^{N-1} x[n] e^{-jw_k n} \quad k = 0,1,\ldots, M - 1
\]

with \( W \) defined as

\[
W = e^{-j\Delta w}
\]

we have

\[
X(e^{jw_k}) = \sum_{n=0}^{N-1} x[n] e^{-jw_0n} W^{nk} \quad k = 0,1,\ldots, M - 1
\]

The Chirp transform represents \( X(e^{jw_k}) \) as a convolution:

To achieve this purpose, we represent \( nk \) as

\[
nk = (1/2)[n^2 + k^2 - (k - n)^2]
\]
Then, the DTFT value evaluated at \( w_k \) is

\[
X(e^{jw_k}) = \sum_{n=0}^{N-1} x[n]e^{-jw_0n}W^{n^2/2}W^{-k^2/2}W^{-(k-n)^2/2}
\]

Letting \( g[n] = x[n]e^{-jw_0n}W^{n^2/2} \), we can then write

\[
X(e^{jw_k}) = W^{k^2/2} \left( \sum_{n=0}^{N-1} g[n]W^{-(k-n)^2/2} \right) \quad k = 0,1,\ldots,M-1
\]

To interpret the above equation, we obtain more familiar notation by replacing \( k \) by \( n \) and \( n \) by \( k \):

\[
X(e^{jw_n}) = W^{n^2/2} \left( \sum_{k=0}^{N-1} g[k]W^{-(n-k)^2/2} \right) \quad n = 0,1,\ldots,M-1
\]

\( X(e^{jw_k}) \) corresponds to the convolution of the sequence \( g[n] \) with the sequence \( W^{-n^2/2} \).
The block diagram of the chirp transform algorithm is

\[ x[n] \quad \times \quad \frac{W^{-n^2/2}}{e^{-j\omega_0 n W^{n^2/2}}} \quad \times \quad \frac{W^{n^2/2}}{X(e^{j\omega n})} \]

Since only the outputs of \( n=0,1,...,M-1 \) are required, let \( h[n] \) be the following impulse response with finite length (FIR filter):

\[
h[n] = \begin{cases} 
W^{-n^2/2} & -(N - 1) \leq n \leq M - 1 \\
0 & \text{otherwise}
\end{cases}
\]

Then

\[
X(e^{j\omega n}) = W^{n^2/2} (g[n] * h[n]) \quad n = 0,1,...,M - 1
\]
The block diagram of the chirp transform algorithm for FIR is

\[
\begin{align*}
\times & \quad x[n] \quad g[n] \\
\rightarrow & \quad h[n] \\
\rightarrow & \quad \times \quad y[n] \\
& \quad e^{-j\omega_0 n} W^{n^2/2} \\
& \quad W^{n^2/2}
\end{align*}
\]

Then the output \( y[n] \) satisfies that

\[
X(e^{j\omega_0 n}) = y[n] \quad n = 0, 1, ..., M - 1
\]

Evaluating frequency responses using the procedure of chirp transform has a number of potential advantages:

- We do not require \( N=M \) as in the FFT algorithms, and neither \( N \) nor \( M \) need be composite numbers. => The frequency values can be evaluated in a more flexible manner.
- The convolution involved in the chirp transform can still be implemented efficiently using an FFT algorithm. The FFT size must be no smaller than \( (M+N-1) \). It can be chosen, for example, to be an appropriate power of 2.
In the above, the FIR filter $h[n]$ is non-causal. For certain real-time implementation it must be modified to obtain a causal system. Since $h[n]$ is of finite duration, this modification is easily accomplished by delaying $h[n]$ by $(N-1)$ to obtain a causal impulse response:

$$h_1[n] = \begin{cases} W^{-(n-N+1)^2/2} & n = 0, 1, ..., M + N - 2 \\ 0 & \text{otherwise} \end{cases}$$

and the DTFT transform values are

$$X(e^{j\omega_0 n}) = y_1[n + N - 1] \quad n = 0, 1, ..., M - 1$$

In hardware implementation, a fixed and pre-specified causal FIR can be implemented by certain technologies, such as charge-coupled devices (CCD) and surface acoustic wave (SAW) devices.
We illustrate this topic by an example from musical signal processing.

Suppose we are given a recording of, say, the fourth Symphony of Brahms, as a digitized waveform. We want to perform spectral analysis of the audio signal.

Why would we want to do that? For the sake of the story, assume we want to use the signal for reconstructing the full score of the music, note by note, instrument by instrument.

Let us do a few preliminary calculations. The music is over 40 minutes long, or about 2500 seconds. Compact-disc recordings are sampled at 44.1 kHz and are in stereo.
However, to simplify matters, assume we combine the two channels into a single monophonic signal by summing them at each time point.

Our discrete-time signal then has a number of samples $N$ on the order of $10^8$.

So, are we to compute the DFT of a sequence one hundred million samples long? This appears to be both unrealistic and useless.

It is unrealistic because speed and storage requirements for a hundred million point FIT are too high by today's standards.

It is useless because all we will get as a result is a wide-band spectrum, including all notes of all instruments, with their harmonics, throughout the symphony.
If not a full-length DFT, then what?

A bit of reflection will tell us that what we really want is a sequence of short DFTs. Each DFT will exhibit the spectrum of the signal during a relatively short interval.

Thus, for example, if the violins play the note E during a certain time interval, the spectrum should exhibit energies at the frequency of the note E (329.6 Hz) and at the characteristic harmonics of the violin.

In general, if the intervals are short enough, we may be able to track the note-to-note changes in the music. If the frequency resolution is good enough, we may be able to identify the musical instrument(s) playing at each time interval.

This is the essence of spectral analysis.

The human ear-brain is certainly an excellent spectrum analyzer. A trained musician can identify individual notes and instruments in very complex musical compositions.
Continuing our example, each DFT would have a length of, say, 4096, corresponding to a time interval of about 92.9 milliseconds.

The frequency resolution will be about 11 Hz, which is quite adequate.

There are about 26,000 distinct intervals in the symphony[= \((40 \times 60)/(92.9 \times 10^{-3})\)], so we will need to compute 26,000 FFTs of length 4096.

We may choose to be conservative and overlap the intervals to a certain extent (say 50 percent), to ensure that nothing important that may occur at the border between adjacent intervals is missed.

Then we will need about 52,000 FFTs of length 4096. With today's technology, the entire computation can be done in considerably less time than the music itself.
Dividing a signal having long duration into short segments and performing spectral analysis on each segment is known as short-time spectral analysis.

Practical short-time spectral analysis requires more than just DFT (or FFT) computations.

We study the principal technique used for short-time spectral analysis, that of *windowing*.

We first describe the basic windowing operation and interpret it in the time and frequency domains.

We then list the most common windows and examine their properties.

Finally, we apply the windowing technique to the problem of measuring the frequencies of sinusoidal signals, first without and then with additive noise.
Figure 6.1 Brahms's Symphony No. 4, fourth movement, bars 1 through 4 (top to bottom), a 92.9-millisecond segment of each bar.
Figure 6.3 Spectra of the signals shown in Figure 6.1. x’s denote frequencies of chord notes and their harmonics.
Windowing

The effect of rectangular windowing

- Assume we are given a long, possibly infinite-duration signal $y[n]$. We pick a relatively short segment of $y[n]$, say

$$x[n] = \begin{cases} y[n], & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise.} \end{cases}$$

- We can describe the operation of getting $x[n]$ from $y[n]$ as a multiplication of $y[n]$ by a rectangular window, that is,

$$x[n] = y[n]w_r[n], \quad \text{where} \quad w_r[n] = \begin{cases} 1, & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise.} \end{cases}$$
Recall that multiplication in the time domain translates to convolution in the frequency domain, so

\[ X^f(\theta) = \frac{1}{2\pi} \{ Y^f \ast W^f_r \}(\theta), \]

where \( W^f_r(\theta) \) is the Fourier transform of the rectangular window, given by

\[ W^f_r(\theta) = \sum_{n=0}^{N-1} e^{-j\theta n} = \frac{1 - e^{-j\theta N}}{1 - e^{-j\theta}} = \frac{\sin(0.5\theta N)}{\sin(0.5\theta)} e^{-j0.5\theta(N-1)}. \]

The function

\[ D(\theta, N) = \frac{\sin(0.5\theta N)}{\sin(0.5\theta)} \]

is called the Dirichlet kernel.

Note that this kernel has also appeared in the constructing DTFT from DFT for a final-length signal.
Figure 6.6 The Dirichlet kernel for $N = 40$. 
Main property of the Dirichlet kernel are as follows

- Its maximum value is \( N \), occurring at \( \theta = 0 \).
- Its zeros nearest to the origin are at \( \theta = \pm 2\pi/N \). The region between these two zeros is called the **main lobe** of the Dirichlet kernel.
- There are additional zeros at \( \{ \theta = 2m\pi/N \mid m = \pm 2, \pm 3, \ldots \} \).
- Between every pair of adjacent zeros there is a local maximum or a local minimum, approximately at \( \theta = (2m + 1)\pi/N \). The regions between these zeros are called **side lobes**.
- The highest side lobe (in absolute value) occurs at \( \theta = \pm 3\pi/N \) and its value (for large \( N \)) is approximately \( 2N/3\pi \). The ratio of the highest side lobe to the main or about \(-13.5 \text{ dB}\).
What is the ideal kernel?

- Its shall be an impulse function between a single period $[-\pi, \pi]$.

- i.e., it forms an impulse train in the DTFT domain, which implies the ideal case that the time domain is 1 for all $n$.

- Rectangular windowing has undesirable side effects, so it is generally not a good way of performing short-time spectral analysis.
Common windows

- As we have seen, the undesirable side effects of rectangular windowing are due to the convolution of the Fourier transform of the original signal with the Dirichlet kernel.
- Suppose, instead, that we wish to obtain $X_f(\theta)$ as the convolution of $Y_f(\theta)$ with another function $W_f(\theta)$.
- The time-domain multiplication property of the Fourier transform tells us how to do it: We must multiply $y[n]$ by the sequence $w[n]$, the inverse Fourier transform of $W_f(\theta)$. Thus, we must perform the operation

$$x[n] = y[n]w[n],$$

and then we will get

$$X_f(\theta) = \frac{1}{2\pi} \{Y_f \ast W_f\}(\theta).$$
The sequence $w[n]$ is called a window, and the operation is called windowing.

Windowing amounts to truncating the signal $y[n]$ to a finite length $N$ while reshaping it through multiplication.

Choosing a window always involves trade-off between the width of the main lobe and the level of the side lobes.

In general, the narrower the main lobe, the higher the side lobes, and vice versa.

Property of rectangular window: the rectangular has the narrowest possible main lobe of all windows of the same length, but its side lobes are the highest.

The side-lobe level of the rectangular window, which is $-13.5$ dB, is undesired in most applications.
Usually, the frequency is drawn in the log domain.

For rectangular window:

Figure 6.8 A rectangular window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.
Bartlett Window

- Bartlett’s method of side-lobe level reduction is based on squaring of the kernel function of the triangular window, thereby reducing the side-lobe level by a factor of 2 (in dB).
- Suppose that the desired window length $N$ is odd.
- Let $w_r[n]$ be a rectangular window of length $(N+1)/2$. Define the Bartlett window $w_t[n]$ from the rectangular window as

$$
w_t[n] = \frac{2}{N+1} \{w_r * w_r\}[n] = 1 - \frac{|2n - N + 1|}{N+1}, \quad 0 \leq n \leq N-1.
$$

- Since convolving a rectangular window with itself in the time domain is equivalent to squaring in the time domain, the corresponding kernel function is then

$$W_t^f(\theta) = \frac{2}{N+1} D^2(\theta, 0.5(N + 1))e^{-j0.5\theta(N-1)} = \frac{2 \sin^2[0.25\theta(N + 1)]}{(N + 1) \sin^2(0.5\theta)} e^{-j0.5\theta(N-1)}. \quad (6.11)$$
Figure 6.9 Bartlett window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.
This window is called the Bartlett window (after its discoverer) or a triangular window (owing to its shape).

As from its construction, the side-lobe level of the Bartlett window is $-27$ dB. The width of the main lobe is $8\pi/(N + 1)$, which is nearly twice that of a rectangular window of the same length.

When $N$ is even, the Bartlett window is defined as

$$W_t^f(\theta) = \frac{2}{(N + 1)} D(\theta, 0.5N) D(\theta, 0.5(N + 2)) e^{-j0.5\theta(N-1)}$$

$$= \frac{2 \sin(0.25\theta N) \sin[0.25\theta(N + 2)]}{(N + 1) \sin^2(0.5\theta)} e^{-j0.5\theta(N-1)}.$$
Hann Window (or Hanning window)

- Whereas the Bartlett window achieves side-lobe level reduction by squaring, the Hann window achieves a similar effect by superposition.

- The kernel function of the Hann window is obtained by adding three Dirichlet kernels, shifted in frequency so as to yield partial cancellation of their side lobes.

Figure 6.10 Construction of the Hann window from three Dirichlet kernels. Dashed line: center kernel; dot-dashed lines: shifted kernels; solid line: the sum.
The kernel function of Hann window in the frequency is given by

\[ W_{hn}^f(\theta) = \left[ 0.5D(\theta, N) + 0.25D\left(\theta - \frac{2\pi}{N-1}, N\right) + 0.25D\left(\theta + \frac{2\pi}{N-1}, N\right) \right] e^{-j0.5\theta(N-1)} \]

\[ = 0.5W_r^f(\theta) - 0.25W_r^f\left(\theta - \frac{2\pi}{N-1}\right) - 0.25W_r^f\left(\theta + \frac{2\pi}{N-1}\right). \]  

(6.13)

Its window function in time domain is

\[ w_{hn}[n] = 0.5 - 0.25 \exp\left(\frac{j2\pi n}{N-1}\right) - 0.25 \exp\left(-\frac{j2\pi n}{N-1}\right) \]

\[ = 0.5 \left[ 1 - \cos\left(\frac{2\pi n}{N-1}\right) \right], \quad 0 \leq n \leq N - 1. \]

The Hann window is also called the cosine window.

The side-lobe level of this window is \(-32\) dB and the width of the main lobe is \(8\pi/N\).
Figure 6.11 Hann window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.
The Hann window has a peculiar property: Its two end points are zero. When applied to a signal $y[n]$, it effectively deletes the points $y[0]$ and $y[N-1]$. This suggests increasing the window length by 2 with respect to the desired $N$ and delete the two end points. The modified Hann window thus obtained is

$$w_{hn}[n] = 0.5 \left\{ 1 - \cos \left( \frac{2\pi(n + 1)}{N + 1} \right) \right\}, \quad 0 \leq n \leq N - 1;$$
Hamming Window

- Obtained by a slight modification of the Hann window, which amounts to choosing different magnitudes for the three Dirichlet kernels.

\[
\begin{align*}
    w_{hm}[n] &= 0.54 - 0.46 \cos \left( \frac{2\pi n}{N - 1} \right), \quad 0 \leq n \leq N - 1, \\
    W_{hm}^f(\theta) &= 0.54W_r^f(\theta) - 0.23W_r^f \left( \theta - \frac{2\pi}{N - 1} \right) - 0.23W_r^f \left( \theta + \frac{2\pi}{N - 1} \right).
\end{align*}
\]

Figure 6.12  Construction of the Hamming window from three Dirichlet kernels. Dashed line: center kernel; dot-dashed lines: shifted kernels; solid line: the sum.
Figure 6.13 Hamming window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.
As we see, the side lobes of the sum are lower than those of the Hann window.

Hamming got the numbers 0.54 and 0.46 by trial and error, seeking to minimize the amplitude of the highest side lobe.

The side-lobe level of this window is $-43$ dB, and the width of the main lobe is $8\pi/N$.

The Hamming window is also called the raised-cosine window.
**Blackman Window**

- The Hamming window has the lowest possible side-lobe level among all windows based on three Dirichlet kernels.
- The Blackman window uses five Dirichlet kernels, thus reducing the side-lobe level still further.

\[ w_b[n] = 0.42 - 0.5 \cos \left( \frac{2\pi n}{N - 1} \right) + 0.08 \cos \left( \frac{4\pi n}{N - 1} \right), \quad 0 \leq n \leq N - 1, \]

\[ W_b^f(\theta) = 0.42W_r^f(\theta) - 0.25W_r^f \left( \theta + \frac{2\pi}{N - 1} \right) - 0.25W_r^f \left( \theta - \frac{2\pi}{N - 1} \right) \\
+ 0.04W_r^f \left( \theta + \frac{4\pi}{N - 1} \right) + 0.04W_r^f \left( \theta - \frac{4\pi}{N - 1} \right). \]

- The side-lobe level of the Blackman window is \(-57 \text{ dB}\) and the width of the main lobe is \(12\pi/N\).
- As in the case of the Hann window, the two end points of the Blackman window are zero, so in practice we can increase \(N\) by 2 and remove the two end points.
Figure 6.14 Blackman window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.
Kaiser Window

- The windows described so far are considered as classical. They have been derived based on intuition and educated guesses.
- Modern windows are based on optimality criteria; they aim to be best in certain respect, while meeting certain constraints.
- Different optimality criteria give rise to different windows.
- Dolph’s criterion: Minimize the width of the main lobe of the kernel, under the constraint that the window length is fixed and the side-lobe level not exceed a given maximum value.
Kaiser criterion: Minimize the width of the main lobe of the kernel, under the constraint that the window length be fixed and the energy in the side lobes not exceed a given percentage of the total energy. The energy in the side lobes is defined as the integral of the square magnitude of the kernel function over the range \([-\pi, \pi]\), excluding the interval of the main lobe.

Kaiser window:

- Of the windows based on these two criteria, the Kaiser window is much more popular than Dolph window.
- Kaiser criterion gives rise to a family of windows that has become, perhaps, the most widely used for modern digital signal processing.
The solution of Kaiser’s optimization problem is described in terms of the modified Bessel function of order zero. This function is given by the infinite power series

\[ I_0(x) = \sum_{k=0}^{\infty} \left( \frac{x^k}{2^k k!} \right)^2. \]

Using this function, the Kaiser window is given by

\[ w_k[n] = \frac{I_0\left[ \alpha \sqrt{1 - \left( \frac{|2n-N+1|}{N-1} \right)^2} \right]}{I_0[\alpha]}, \quad 0 \leq n \leq N-1. \]

The parameter \( \alpha \) is used for controlling the main-lobe width and the side-lobe level.

Higher \( \alpha \) leads to a wider main lobe and lower side lobes.
Figure 6.15 Properties of the Kaiser window as a function of the parameter $\alpha$: (a) main-lobe width, as a multiple of $2\pi/N$; (b) side-lobe level.
The following figure depicts the Kaisor window for $N=41$ and $\alpha=12$. In this case the main-lobe width is $16\pi/N$ and the side-lobe level is $-90\text{dB}$.

![Graph](image)

Figure 6.16 Kaiser window, $N = 41$, $\alpha = 12$: (a) time-domain plot; (b) frequency-domain magnitude plot.
Frequency Measurement

- One of the most important applications of the DFT is the measurement of frequencies of periodic signals (in particular sinusoidal signals), which is essential to spectral analysis.
- Practical signals are measured over a finite time interval and the Fourier transform can be computed only on a discrete set of frequencies.
- It is convenient to deal with complex exponential signals first, and proceed to real-valued sinusoids later.
  - In practice, real-valued sinusoids are much common. However, in certain applications (such as radar and communication), complex signals appear naturally.
The simplest case of a signal for which frequency measurement is a meaningful problem is a single complex exponential:

\[ y(t) = A e^{j(\omega_0 t + \phi_0)}, \]

and we wish to measure the frequency \( \omega_0 \).

We sample the signal at interval \( T \) and collect \( N \) consecutive data points, thus getting the discrete-time signal

\[ y[n] = A e^{j(\theta_0 n + \phi_0)}, \quad 0 \leq n \leq N - 1, \]

where \( \theta_0 = \omega_0 T \). We assume that \(-\pi \leq \omega_0 T \leq \pi\), so measurement of \( \theta_0 \) implies unambiguous measurement of \( \omega_0 \).
The DTFT of the sampled signal is given by

$$Y^f(\theta) = Ae^{j\phi_0} \sum_{n=0}^{N-1} e^{-j(\theta-\theta_0)n}$$

$$= Ae^{-j[0.5(\theta-\theta_0)(N-1)-\phi_0]}D(\theta - \theta_0, N).$$

where $D(.,.)$ is the Dirichlet kernel.

In particular, $D(0,N)=N$, evaluation of the DTFT at the frequency $\theta = \theta_0$ gives

$$Y^f(\theta_0) = N Ae^{j\phi_0}, \text{ therefore } |Y^f(\theta_0)| = NA.$$

Furthermore, since $|D(\theta,N)| < N$ for all $\theta \neq \theta_0$, the point $\theta = \theta_0$ is the unique global maximum of $|Y^f(\theta)|$ on $-\pi \leq \theta \leq \pi$. We therefore conclude that $\theta_0$ can be obtained by finding the point of global maximum of $|Y^f(\theta)|$ in the frequency $(-\pi,\pi)$. 
To reiterate

The magnitude of the Fourier transform of a finite segment of a complex exponential signal \( y[n] = Ae^{j(\theta_0 n + \phi_0)} \) exhibits a unique global maximum at the frequency \( \theta = \theta_0 \), thus enabling the determination of \( \theta_0 \) from \( |Y_f(\theta)| \).

In practice, it is not possible to find the global maximum of \( |Y_f(\theta)| \) exactly, since we cannot evaluate this function at the infinite number of frequency points.

A first approximation can be obtained by computing the DFT of \( y[n] \) and searching for the point of maximum of \( Y^d(k) \), the DFT of \( y[n] \).

The index \( k_0 \) for which \( Y^d(k) \) is maximized yields a corresponding frequency \( 2\pi k_0 / N \).

Better approximation can be obtained if necessary, by either zero padding or using a Chirp transform to increase the resolution of sampling in the DTFT domain.
Frequency measurement for two complex exponentials

Proceeding to a more difficult problem, we now consider a signal consisting of two complex exponentials:

\[ y(t) = A_1 e^{j(\omega_1 t + \phi_1)} + A_2 e^{j(\omega_2 t + \phi_2)}. \]

Our aim is to measure the frequencies \( \omega_1 \) and \( \omega_2 \). As before, we sample the signal at interval \( T \) and collect \( N \) measurements of the sampled signal to obtain

\[ y[n] = A_1 e^{j(\theta_1 n + \phi_1)} + A_2 e^{j(\theta_2 n + \phi_2)}, \quad 0 \leq n \leq N - 1, \]

where \( \theta_i = \omega_i T, \ i = 1, 2; \ we \ assume \ that \ -\pi < \omega_i T < \pi. \]
The Fourier transform of the sampled signal is given by

\[ Y^f(\theta) = A_1 e^{j\phi_1} \sum_{n=0}^{N-1} e^{-j(\theta-\theta_1)n} + A_2 e^{j\phi_2} \sum_{n=0}^{N-1} e^{-j(\theta-\theta_2)n} \]

\[ = A_1 e^{-j[0.5(\theta-\theta_1)(N-1)-\phi_1]} D(\theta - \theta_1, N) \]
\[ + A_2 e^{-j[0.5(\theta-\theta_2)(N-1)-\phi_2]} D(\theta - \theta_2, N). \]

In particular, evaluation of the Fourier transform at the frequency \( \theta = \theta_1 \) gives

\[ Y^f(\theta_1) = NA_1 e^{j\phi_1} + A_2 e^{-j[0.5(\theta_1-\theta_2)(N-1)-\phi_2]} D(\theta_1 - \theta_2, N). \]

We observe the following:
1. If $A_2 = 0$, that is, if only one complex sinusoid is present, then $|Y^f(\theta_1)| = NA_1$, as before. Therefore we can find $\theta_1$ from the maximum point of $|Y^f(\theta)|$.

2. If $A_2 \neq 0$, but

$$|A_2 D(\theta_1 - \theta_2, N)| \ll NA_1,$$  \hspace{1cm} (6.32)

we still have a good chance of finding a local maximum of $|Y^f(\theta)|$ near $\theta_1$, since the behavior of $|Y^f(\theta)|$ near $\theta_1$ will not be much perturbed by the second term in (6.31). However, this local maximum is not necessarily a global maximum any more, because the second term in (6.31) affects the global behavior of $|Y^f(\theta)|$.

3. Condition (6.32) will hold if $|\theta_2 - \theta_1| \geq 2\pi / N$, and if $A_2$ is not much larger than $A_1$. Otherwise, (6.32) may fail, and we may be unable to find a local maximum of $|Y^f(\theta)|$ near $\theta_1$.

4. The discussion is symmetric with respect to the two sinusoidal components. We can find another local maximum of $|Y^f(\theta)|$ near $\theta_2$, provided the interference from the Dirichlet kernel centered at $\theta_1$ is sufficiently small in the vicinity of $\theta_2$. 
Example 6.1 Figure 6.18 shows the behavior of $|Y^f(\theta)|$ in the range of $\theta$ of interest (in dB below the maximum) for several choices of $\theta_1$, $\theta_2$, $A_1$, and $A_2$ (in all cases $N = 64$). In parts a, b, and c the amplitudes of the two complex exponentials are equal. In part a, the difference $\theta_2 - \theta_1$ is $2\pi/N$. As we see, the two peaks are well separated, indicating the presence of two complex exponentials. However, the peaks appear to repel each other, causing their locations to deviate slightly from the true $\theta_1$ and $\theta_2$. This phenomenon is called bias; it typically occurs when the frequencies are close to each other. In part b, the frequency difference is $1.5\pi/N$. Now the two peaks start to merge with each other, but are still distinct; bias is again apparent. In part c, the frequency difference is $\pi/N$. In this case there is only one peak, so the two complex exponentials cannot be distinguished from each other.
Figure 6.18 The Fourier transform of two complex exponentials in Example 6.1: (a) $A_2 = A_1$, $\theta_2 - \theta_1 = 2\pi/N$; (b) $A_2 = A_1$, $\theta_2 - \theta_1 = 1.5\pi/N$; (c) $A_2 = A_1$, $\theta_2 - \theta_1 = \pi/N$; (d) $A_2 = 0.25A_1$, $\theta_2 - \theta_1 = 2\pi/N$; (e) $A_2 = 0.25A_1$, $\theta_2 - \theta_1 = 1.5\pi/N$; (f) $A_2 = 0.25A_1$, $\theta_2 - \theta_1 = \pi/N$. 
Refinement by windowing

Condition (6.32) may be hard to satisfy since the side lobes of the Dirichlet kernel are relatively high; the highest side lobe is only 13.5 dB lower than the main lobe. For example, if $|\theta_2 - \theta_1| = 3\pi/N$ and $A_2$ is larger than $A_1$ by more than 13.5 dB, the Dirichlet kernel centered at $\theta_2$ will interfere with the one centered at $\theta_1$ and may prohibit measurement of $\theta_1$. Because of the slow rate of decay of the side lobes of the Dirichlet kernel, however, the problem does not completely disappear, even if the distance between the two frequencies is large with respect to $2\pi/N$.

Windowing alleviates the problems we have described. Suppose we multiply $y[n]$ by a window $w[n]$ prior to computing the Fourier transform and denote

$$x[n] = y[n]w[n].$$

Then (6.30) changes to

$$X^f(\theta) = A_1 e^{j\phi_1} \sum_{n=0}^{N-1} w[n] e^{-j(\theta - \theta_1)n} + A_2 e^{j\phi_2} \sum_{n=0}^{N-1} w[n] e^{-j(\theta - \theta_2)n}$$

$$= A_1 e^{j\phi_1} W^f(\theta - \theta_1, N) + A_2 e^{j\phi_2} W^f(\theta - \theta_2, N),$$

(6.33)
We can increase the chances of meeting the later condition.

For example, if we use a Kaiser window with side-lobe level of $-80\text{dB}$, we will be able to handle sinusoidal components whose amplitudes differ by up to four orders of magnitude.

However, this comes at the price of making the frequency separation condition more difficult to meet, due to the widening of the main lobe.

\(\mathcal{W}^f(\theta, N)\) is the kernel function of the window. In particular,

\[
X^f(\theta_1) = A_1 e^{j\phi_1} W^f(0, N) + A_2 e^{j\phi_2} W^f(\theta_1 - \theta_2, N). \tag{6.34}
\]

We have \(W^f(0, N) = \sum_{n=0}^{N-1} w[n]\), and this sum is approximately proportional to \(N\). Suppose that

\[
|A_2 W^f(\theta_1 - \theta_2, N)| \ll A_1 \sum_{n=0}^{N-1} w[n]; \tag{6.35}
\]

then we have a good chance of finding a local maximum of \(|X^f(\theta)|\) near \(\theta_1\). \textbf{Condition} (6.35) holds if \(|\theta_2 - \theta_1|\) is greater than the width of the main lobe of the kernel function, and if \(20 \log_{10}(A_1/A_2)\) is larger than the side-lobe level.
Example 6.2 Figure 6.19 shows the behavior of the windowed DFT $|X^f(\theta)|$ for two complex exponentials, using the Hann window. The amplitudes $A_1, A_2$ are the same as in Example 6.1. The frequency differences are $8\pi/N, 6\pi/N,$ and $4\pi/N$. As we see, the two peaks are well separated when the frequency difference is $8\pi/N$, and are still distinguishable when it is $6\pi/N$. The bias is considerably smaller than in the case of unwindowed DFT. However, the two complex exponentials become indistinguishable when the frequency difference is $4\pi/N$, whereas with unwindowed DFT they would be well separated (as we recall from Example 6.1).
Figure 6.19 The windowed Fourier transform of two complex exponentials in Example 6.2: (a) $A_2 = A_1$, $\theta_2 - \theta_1 = 8\pi/N$; (b) $A_2 = A_1$, $\theta_2 - \theta_1 = 6\pi/N$; (c) $A_2 = A_1$, $\theta_2 - \theta_1 = 4\pi/N$; (d) $A_2 = 0.25A_1$, $\theta_2 - \theta_1 = 8\pi/N$; (e) $A_2 = 0.25A_1$, $\theta_2 - \theta_1 = 6\pi/N$; (f) $A_2 = 0.25A_1$, $\theta_2 - \theta_1 = 4\pi/N$. 
Practice of frequency measurement in spectral analysis

- Based on the above studies, practical frequency measurement using DFT consists, as a minimum, of the following three steps:
  1. Multiplication of the sampled sequence by a window.
  2. Computation of the DFT, usually through FIT.
  3. Search for the local maxima of the absolute value of the DFT and selection of the maxima of interest.

- Additional steps are necessary if measurement of the amplitudes and phases is required as well, but we shall not discuss them here.

- As we have explained, the choice of a window requires knowledge of the nature of the signal.
1. Suppose that a computer program is available for computing the DFT

\[ X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}, \quad k = 0,1,\ldots, N-1 \]

i.e., the input to the program is the sequence \( x[n] \) and the output is the DFT \( X[k] \). Show how the input and/or output sequences may be rearranged such that the program can also be used to compute the inverse DFT

\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}, \quad n = 0,1,\ldots, N-1 \]
2. Consider the systems shown in the following figure. Suppose that $H_1(e^{jw})$ is fixed and known. Find $H_2(e^{jw})$, the frequency response of an LTI system, such that $y_2[n] = y_1[n]$ if the inputs to the systems are the same.
3. Discuss whether windowing is needed for frequency measurement of a single real sinusoid.